Math Club Notes: 2005 June 27

## 1 Compositions

Yet another tidbit from Cmpt 272:

> A *composition* of a positive integer $n$ is a way to write $n$ as a sum of positive integers. Order matters; e.g., "$1+1+2$" and "$1+2+1$" are considered distinct compositions of 4. A "sum" consisting of just one term is also considered a composition, so that, for example, "4" is a composition of 4.
>
> How many compositions of $n$ are there? That is, give a closed-form expression for the number of compositions of $n$ as a function of $n$, and prove it correct.

(This way of posing the question is unusually explicit; for example, the final sentence, which explains what kind of answer is expected, would often be omitted. To a mathematician, what that sentence says, er, goes without saying.)

Mehran, following general principles, suggested that we look at the first few values. "When in doubt, use brute force." Let $C_n$ denote the number of compositions of $n$. We compute:

| $n$ | Compositions | $C_n$ |
|---|---|---|
| 1 | $1$ | 1 |
| 2 | $2$ | 2 |
| | $1+1$ | |
| 3 | $3$ | 4 |
| | $2+1$ | |
| | $1+2$ | |
| | $1+1+1$ | |
| 4 | $4$ | 8 |
| | $3+1$ | |
| | $1+3$ | |
| | $2+2$ | |
| | $2+1+1$ | |
| | $1+2+1$ | |
| | $1+1+2$ | |
| | $1+1+1+1$ | |

Hey — powers of 2.

Conjecture:

$$\forall n \in \mathbb{Z}^+ \colon C_n = 2^{n-1}\ .$$

This is a statement about all positive integers; induction should leap to mind as a standard way to prove such things.

### 1.1 Proof by induction

We already have the base case (and then some) in the table above; what about the inductive step? We will suppose our result true for $n$ and then show it follows for $n+1$. That is, our situation in the inductive step is:

$$\begin{aligned} C_n &= 2^{n-1} && \text{(supposed)} \\ C_{n+1} &= 2^n && \text{(desired)} \end{aligned}$$

Trying to make what we've supposed look more like what we want to show, we multiply the inductive hypothesis by 2:

$$\begin{aligned} 2C_n &= 2^n && \text{(supposed)} \\ C_{n+1} &= 2^n && \text{(desired)} \end{aligned}$$

If only we could show that $C_{n+1} = 2C_n$. Then we would have a proof, which would look like this:

> We show by induction that $C_n = 2^{n-1}$ for all positive integers $n$. In the base case $n = 1$, we have $C_1 = 1$ (since there is only composition of 1, namely "1"), and $2^{1-1} = 1$. Now suppose the claim to be true for some positive integer $k$, that is, $C_k = 2^{k-1}$. Then
>
> $$\begin{aligned} C_{k+1} &= 2C_k && \text{(by magic)} \\ &= 2 \cdot 2^{k-1} && \text{(inductive hypothesis)} \\ &= 2^k \end{aligned}$$
>
> which is exactly the claim for $k+1$. By induction, the claim holds for all positive integers $n$.

Considerable progress. By deploying the standard technique of induction, we've reduced our problem from proving the closed form for our sequence to proving a relationship between consecutive elements of that sequence: it remains only to show that $C_{n+1} = 2C_n$. It's natural now to try to separate the $C_{n+1}$ compositions of $n+1$ into two groups, in such a way that both groups are obviously as numerous as the compositions of $n$.

One way to get such a group is pretty easy to think of: take a composition of 3 (for concreteness), tack on a "+1", and you'll have a composition of 4.

$$\left.\begin{aligned} 3 &\to 3+1 \\ 2+1 &\to 2+1+1 \\ 1+2 &\to 1+2+1 \\ 1+1+1 &\to 1+1+1+1 \end{aligned}\right\} \quad \text{Group A: append "+1"}$$

Now we have to figure out a rule for the other group:

$$\begin{aligned} 3 &\to 4 \\ 2+1 &\to 2+2 \\ 1+2 &\to 1+3 \\ 1+1+1 &\to 1+1+2 \end{aligned} \qquad \text{Group B: what rule?}$$

Ray spotted the pattern here: add 1 to the last term in the composition. Obviously this turns a composition of 3 into a composition of 4.

Putting it all together:

**Lemma 1** $C_{n+1} = 2C_n$ for all positive integers $n$.

*Proof* Divide the compositions of $n+1$ into two sets: set $A$ containing those compositions whose last term is 1, and set $B$ containing those whose last term is greater than 1. Obviously these sets are disjoint, so that $C_{n+1} = |A| + |B|$.

Set $A$ is equipollent with the compositions of $n$, that is, $|A| = C_n$. For each composition of $n+1$ in $A$ is a composition of $n$ with "+1" tacked on the end. (Obviously this is a bijection.)

Also, $|B| = C_n$, since each composition of $n+1$ in $B$ is a composition of $n$ with the last term increased by 1. (This too is obviously a bijection.)

Therefore $C_{n+1} = |A| + |B| = 2C_n$. □

Then we give the inductive proof from before (replacing "by magic" with "by the lemma"), and we're done.

### 1.2 Luck

You might have noticed that, when figuring out group B, we wrote down

$$\begin{aligned} 3 &\to 4 \\ 2+1 &\to 2+2 \\ 1+2 &\to 1+3 \\ 1+1+1 &\to 1+1+2 \end{aligned}$$

By luck, it seems, this turned out to be the right way to pair up the compositions of 3 with the compositions in group B. But how would we know that?

The only thing I can think of is, try to pair them up by "natural" similarity. We might notice that, of the compositions we wish to pair up, on both sides we have one of length 1, two of length 2, and one of length 3. It's reasonable to guess that we should pair them up by length. The two of length 2 can be paired correctly by matching the common first elements. Try to make patterns, or bring out patterns that are already there, however vague or partial. It's all guesswork, but that's math.

(That last comment might seem a little odd. Let me put it this way: the product of mathematics — the theorems and proofs — is logical, structured, more-or-less rigorous argument. The process of doing mathematics, however, is intuitive, unstructured, sloppy guesswork. We guess that induction is the way to go, and hope to figure out the inductive step. We guess about how to make group A, and hope we can figure out a rule for group B.)

### 1.3 Brute force

We started with listing all the compositions of $n$ (for the first few $n$) by brute force. One practical difficulty with this plan: how will you know that you've got them all?

The usual idea for this is, try to list them systematically. In the list on page 1, I listed the compositions in order of largest element (sort of). Here's another way: list them lexicographically. The compositions of 4, for example, must each start with 1, 2, 3, or 4. So:

$$1 + \cdots \qquad 2 + \cdots \qquad 3 + \cdots \qquad 4 + \cdots$$

Now, the compositions that start with 1 must then continue with 1, 2, or 3. Not anything larger, lest the sum be larger than 4. So:

$$\begin{array}{llll} 1+1+\cdots & 2+\cdots & 3+\cdots & 4+\cdots \\ 1+2+\cdots & & & \\ 1+3+\cdots & & & \end{array}$$

Proceeding in this manner, we can generate a complete list, and be sure it is complete.

Listing things systematically requires that you come up with a system that will definitely produce everything you want. That in turn requires thinking a little bit about the structure of the set of things you want. This can lead to ideas for proofs; this lexicographic order, for example, suggests grouping the compositions by their first term, which leads to…

### 1.4 Proof by strong induction

We show by strong induction that $C_n = 2^{n-1}$ for all positive integers $n$. The base case $n = 1$ is given above.

Now suppose that, for some integer $k$, the claim is true for all integers $i$ such that $1 \le i < k$. Then count the compositions of $k$ as follows. The first term of each composition must be one of the numbers $1, 2, \dots, k$. There is exactly one composition whose first term is $k$, namely "$k$". If the first term is 1, then the remaining terms form one of the $C_{k-1}$ compositions of $k-1$; if the first term is 2, then the remaining terms form one of the $C_{k-2}$ compositions of $k-2$; and so forth, up to a first term of $k-1$ followed by a composition of 1. Therefore

$$\begin{aligned} C_k &= 1 + C_{k-1} + C_{k-2} + \cdots + C_1 \\ &= 1 + 2^{k-2} + 2^{k-3} + \cdots + 2^{1-1} && \text{(inductive hypothesis)} \\ &= 1 + \frac{2^{k-1}-1}{2-1} && \text{(geometric series)} \\ &= 2^{k-1}, \end{aligned}$$

which is the claim for $k$.

By induction, the claim holds for all positive integers.

### 1.5 Combinatorial proof

The number of compositions of $n$ is clearly the number of ways to chop up a row of $n$ widgets into contiguous blocks, each block containing at least one widget. For example:

$$1+3+2+2 \qquad \square \,|\, \square\,\square\,\square \,|\, \square\,\square \,|\, \square\,\square$$

Consider the $n-1$ interstices between the $n$ widgets:

$$\begin{array}{ccccccccccccccc} & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \square & & \square & & \square & & \square & & \square & & \square & & \square & & \square \end{array}$$

To chop up a row of widgets in the prescribed manner, we may simply choose, for each interstice, whether or not to break the row up at that point. That's $n-1$ independent choices, each among 2 options, so there are $2^{n-1}$ ways to chop up the row.

## 2 Gowers

Perusing Timothy Gowers's website (mentioned in the notes for May 22), I find that he has discussed some of things we have discussed. His approaches are very insightful; I encourage you to look at them.

We discussed the Mean Value Theorem, and interpreted it as a tool for reasoning from tangent slopes to secant slopes. Gowers shows why we need such a theorem by trying to prove that a differentiable function is increasing iff its derivative is nonnegative without using MVT. See http://www.dpmms.cam.ac.uk/~wtg10/meanvalue.html.

We also discussed whether $\mathbb{Z}$ can be bounded above in some larger ring. Gowers approaches the question from a different point of view: he observes that the proof that $\mathbb{Z}$ is unbounded in $\mathbb{R}$ (see the notes for June 6) uses the completeness axiom, and inquires whether this is really necessary. He then proves that it is, by furnishing an example of a structure satisfying all the other axioms of $\mathbb{R}$ (i.e., an ordered field), but in which $\mathbb{Z}$ is bounded. The construction is similar to the one in our notes for June 13, but better. See http://www.dpmms.cam.ac.uk/~wtg10/meta.integers.html. Note that, from Gowers's perspective, this is a proof about proofs: the fact that $\mathbb{Z}$ can be bounded in an ordered field entails that any proof of $\mathbb{Z}$'s unboundedness in $\mathbb{R}$ must invoke (directly or indirectly) the completeness axiom. He gives a few other proofs of this "metamathematical" type; here's another nice one: http://www.dpmms.cam.ac.uk/~wtg10/meta.fta.html. This one shows that any proof of the Fundamental Theorem of Arithmetic must invoke other properties of $\mathbb{Z}$ than the fact that it is an integral domain (i.e., a cancellative, commutative ring with unity).

## 3 Some sums

Ray asked about finding a closed form for

$$S_n = 1 \cdot n + 2(n-1) + 3(n-2) + \cdots + n \cdot 1 .$$

I suggested this:

$$S_n = \sum_{k=1}^{n} k(n-k+1) = (n+1) \sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 .$$

We know how to do the remaining sums. (The final, simplified result is quite tidy; there should be some simple proof of its correctness. I think I might be able to give a geometric proof if I had enough lego.)

On the sums mentioned last week: Ray says he can prove that

$$\sum_{r=1}^{n} \Big( \sum_{s=1}^{r} s \Big)^{-1} = \frac{2n}{n+1} ,$$

and so, as $n \to \infty$, the sum tends to 2. We might look at Ray's proof next week. (The second sum is proving more difficult, as advertised.)

When determining the number of compositions of $n$, and when determining the number of bistrings of length $n$ with no consecutive zeroes (see last week's notes), we applied brute force to get the first few values of a sequence, then recognized the numbers. Sometimes we don't recognize the numbers: for example, consider

$$S_n = \sum_{k} \binom{n}{3k} .$$

Here's the first few values:

| $n$ | Terms | $S_n$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 1 |
| 2 | 1 | 1 |
| 3 | $1+1$ | 2 |
| 4 | $1+4$ | 5 |
| 5 | $1+10$ | 11 |
| 6 | $1+20+1$ | 22 |

Recognize those? I sure don't. (There are still things to notice. For example, each sum is approximately twice the previous one.) I leave the problem of evaluating this sum with you.

## 4 Empty sums and products

We briefly discussed today why an empty sum is $0$ while an empty product is $1$. The crucial fact here is that we're not trying to find out what the "actual" sum with no terms and product with no factors are; we're defining them. "Sum" and "product" are our words. They can mean whatever we want.

So, what do we want? We want our algebraic manipulations to be easy. That is, we want the algebraic rules which apply to sums that actually have terms to apply equally well to sums that don't. That way, we can just apply those manipulations without worrying about special cases like the empty set and whatnot.

For example, let $S$ be some set of integers. It seems natural that

$$\sum_{n \in S} n = \sum_{\substack{n \in S \\ n \text{ even}}} n + \sum_{\substack{n \in S \\ n \text{ odd}}} n \, .$$

Indeed, the proof would be just this statement: every element in $S$ is either even or odd, and cannot be both. When $S$ contains both even and odd elements — say, $S = \{1, 2, 3, 4, 5\}$ — this makes great sense; it says simply that

$$1 + 2 + 3 + 4 + 5 = (2 + 4) + (1 + 3 + 5) \, ,$$

which is an easy consequence of the associativity and commutativity of addition. But what if $S = \{1, 3, 5\}$? Then the statement is

$$1 + 3 + 5 = () + (1 + 3 + 5) \, ,$$

where the $()$ represents a sum of no terms. Obviously the only way this can be true is if that empty sum is $0$.

Similarly, for products it is natural that

$$\prod_{n \in S} n = \Big( \prod_{\substack{n \in S \\ n \text{ even}}} n \Big) \Big( \prod_{\substack{n \in S \\ n \text{ odd}}} n \Big) \, ,$$

and when $S = \{1, 3, 5\}$ this says that

$$1 \cdot 3 \cdot 5 = ()(1 \cdot 3 \cdot 5) \, ,$$

which can only be true if the empty product is $1$.

The way to approach these matters is not with the question, "What is the value of a sum with no terms?" That question presupposes that such a sum has a definite value which we should determine. The right question is, "Is there a way to define 'sums with no terms' so that algebra involving such sums retains the familiar and convenient properties of the algebra of sums that actually have terms?" (And similarly for products.)

As usual, Gowers has a similar, but more profound, discussion; see http://www.dpmms.cam.ac.uk/~wtg10/equations.html for an explanation of how "solving an equation" can be understood as being, in part, the same kind of definition game. Very worth your time. Read it. Now.