Trigonometric motivation for the dot product

The dot product — somewhat bizarre when first encountered — can be well-motivated by the following elementary line of thought.

Suppose we have two vectors in the plane, say, $u = (u_1, u_2)$ and $v = (v_1, v_2)$, and we are interested in the angle θ between them. Thinking of angles, we think of polar coordinates, so we write, say,

$$\begin{aligned} u_1 &= r\cos\alpha & v_1 &= s\cos\beta \\ u_2 &= r\sin\alpha & v_2 &= s\sin\beta \end{aligned} \tag{1}$$

If we know α and β, then we can just subtract:

$$\theta = \beta - \alpha\,. \tag{2}$$

But if we only know the rectangular coordinates u_1, u_2, v_1, v_2, then finding α and β is annoying (and except in a few special cases, computationally difficult with pencil and paper).

Can we still say something about θ, using only the rectangular coordinates? From (1) we see that in the rectangular coordinates we have, not the angles α and β, but trigonometric functions of those angles. Perhaps it would be easier to say something, not about θ, but about some trigonometric function of it. Taking, say, the cosine of both sides of (2) yields

$$\cos\theta = \cos(\beta - \alpha)\,.$$

The only thing to do here is to apply the addition formula for cosine, obtaining

$$\cos\theta = \cos\alpha\cos\beta + \sin\alpha\sin\beta\,.$$

Now the right-hand side involves nearly the same quantities as in (1). We make them exactly the same by multiplying by rs:

$$\begin{aligned} rs\cos\theta &= (r\cos\alpha)(s\cos\beta) + (r\sin\alpha)(s\sin\beta) \\ &= u_1v_1 + u_2v_2\,. \end{aligned}$$

Thus we naturally discover the dot product in $\mathbb{R}^2$. It is then easy to invent the dot product in $\mathbb{R}^n$ by analogy (though not so easy to show that it has the same relation to angle).

Incidentally, if we try for $\sin\theta$ instead of $\cos\theta$, we end up with the determinant. Alas, the higher dimensional analogues are not so obvious here.