On how translating the origin changes norms

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Let $K$ be a symmetric convex body in $\mathbb{R}^n$. (A "body" is a compact set with nonempty interior. By "symmetric" here we mean symmetric about the origin: $K = -K$.) For such $K$ the Minkowski functional

$$\|x\|_K = \inf\{\lambda\colon \lambda > 0 \text{ and } x \in \lambda K\}\,.$$

is a norm and its unit ball is $K$. (When $K$ is not symmetric we continue to think in these terms even though we may have $\|x\|_K \neq \|-x\|_K$.)

In many problems in convex geometry, the choice of origin is somewhat arbitrary; often any point in $\operatorname{int} K$ would do. Thus it is natural to consider how different choices of origin affect the resulting norm. Somewhat more precisely: Let $a \in \operatorname{int} K$ and $p \in \mathbb{R}^n$. Translating to bring $a$ to the origin sends $p$ to $p - a$ and $K$ to $K - a$. How does $\|p\|_K$ relate to $\|p - a\|_{K-a}$?

The maximum and minimum of real numbers $\alpha$ and $\beta$ are denoted $\alpha \wedge \beta$ and $\alpha \vee \beta$ respectively. All other notations are usual ones in convex geometry.

**Proposition 1** If $a \in \operatorname{int} K$, then

$$\|p + a\|_{K+a} \wedge \|p - a\|_{K-a} \geq \|p\|_K$$

for any $p \in \mathbb{R}^n$.

*Proof* Let $\lambda = \|p + a\|_{K+a}$ and $\mu = \|p - a\|_{K-a}$. Then

$$\begin{aligned}
p &= \frac{1}{2}(p + a + p - a) \\
&\in \frac{1}{2}(\lambda(K + a) + \mu(K - a)) && \text{(definition of } \lambda \text{ and } \mu) \\
&= \frac{\lambda + \mu}{2}K + \frac{\lambda - \mu}{2}a \\
&\subseteq \frac{\lambda + \mu}{2}K + \frac{\lambda - \mu}{2}K \\
&= \frac{\lambda + \mu}{2}K + \frac{|\lambda - \mu|}{2}K && (K = -K) \\
&= \left(\frac{\lambda + \mu}{2} + \frac{|\lambda - \mu|}{2}\right)K && \text{(convex distributivity)} \\
&= (\lambda \wedge \mu)K\,.
\end{aligned}$$

Therefore $\|p\|_K \leq \lambda \wedge \mu$, as desired. □

**Proposition 2** If $a \in \operatorname{int} K$ and $p \notin \operatorname{int} K$, then

$$\|p+a\|_{K+a} \vee \|p-a\|_{K-a} \geq \frac{\|p\|_K + \|a\|_K}{1+\|a\|_K} \,.$$

*Proof* Let $\lambda = \|p+a\|_{K+a}$. (Note that $\lambda \geq 1$, since $p+a \notin \operatorname{int}(K+a)$ and $0 \in \operatorname{int}(K+a)$.) Then

$$\begin{aligned}
p &= p+a-a \\
&\in \lambda(K+a)-a && \text{(definition of } \lambda\text{)} \\
&= \lambda K + (\lambda-1)a \\
&\subseteq \lambda K + (\lambda-1)\|a\|_K K && (a \in \|a\|_K K \text{ by definition}) \\
&= (\lambda + (\lambda-1)\|a\|_K)K && \text{(convex distributivity)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\|p\|_K &\leq \lambda + (\lambda-1)\|a\|_K \\
&= \lambda(1+\|a\|_K) - \|a\|_K \,.
\end{aligned}$$

Rearranging yields

$$\frac{\|p\|_K + \|a\|_K}{1+\|a\|_K} \leq \lambda = \|p+a\|_{K+a} \,.$$

Replacing $a$ with $-a$ yields (since $K$ is symmetric, so $\|-a\|_K = \|a\|_K$)

$$\frac{\|p\|_K + \|a\|_K}{1+\|a\|_K} \leq \|p-a\|_{K-a} \,,$$

and together these are the desired result. □

**Proposition 3** If $a \in \operatorname{int} K$ and $p \notin \operatorname{int} K$, then

$$\|p+a\|_{K+a} + \|p-a\|_{K-a} \geq \frac{3}{2}\|p\|_K + \frac{1}{2} \,.$$

*Proof*

$$\begin{aligned}
&\|p+a\|_{K+a} + \|p-a\|_{K-a} \\
&= (\|p+a\|_{K+a} \wedge \|p-a\|_{K-a}) \\
&\qquad + (\|p+a\|_{K+a} \vee \|p-a\|_{K-a}) \\
&\geq \|p\|_K + \frac{\|p\|_K + \|a\|_K}{1+\|a\|_K} && \text{(by the previous propositions)} \\
&= \|p\|_K + \frac{\|p\|_K - 1}{1+\|a\|_K} + 1 \\
&\geq \|p\|_K + \frac{\|p\|_K - 1}{1+1} + 1 && (\|a\|_K < 1 \leq \|p\|_K) \\
&= \frac{3}{2}\|p\|_K + \frac{1}{2} \,.
\end{aligned}$$

□