Integral of powers of sine

We give elementary estimates for, and some standard applications of, the integral

$$I_n = \int_0^\pi \sin^n x \, dx \, .$$

where $n \in \mathbb{N}$.

Proposition 1 For any integer $n \geq 1$,

$$\sqrt{\frac{2\pi}{n+1}} \leq I_n \leq \sqrt{\frac{2\pi}{n}} \, .$$

Proof Integrating by parts yields

$$I_n = \frac{n-1}{n} I_{n-2} \, .$$

It follows, by induction, that

$$I_{n-1} I_n = \frac{2\pi}{n} \, .$$

Since $\sin x \in [0, 1]$ for $x \in [0, \pi]$, the sequence $(I_n)_{n=0}^{\infty}$ is decreasing; thus

$$\frac{2\pi}{n+1} = I_n I_{n+1} \leq I_n^2 \leq I_{n-1} I_n = \frac{2\pi}{n} \, .$$

□

Proposition 2 For any integer $n \geq 2$,

$$I_n = \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \, ,$$

where B_2^n denotes the unit Euclidean ball in $\mathbb{R}^n$.

Proof We compute that

$$\begin{aligned}
\mathrm{vol}_n(B_2^n) &= \int_{-1}^{1} \mathrm{vol}_{n-1}(\sqrt{1-t^2}\, B_2^{n-1}) \, dt \\
&= \int_{-1}^{1} (1-t^2)^{(n-1)/2} \, dt \, \mathrm{vol}_{n-1}(B_2^{n-1}) \\
&= \int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)^{(n-1)/2} \cos\theta \, d\theta \, \mathrm{vol}_{n-1}(B_2^{n-1}) \\
&= \int_{-\pi/2}^{\pi/2} \cos^n\theta \, d\theta \, \mathrm{vol}_{n-1}(B_2^{n-1}) \\
&= I_n \, \mathrm{vol}_{n-1}(B_2^{n-1})
\end{aligned}$$

□

Proposition 3 If $f\colon \mathbb{R}^n \to [0,\infty)$ is positively homogeneous, that is, $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda \in [0,\infty)$, then

$$\int_{S^{n-1}} f(\theta)\, d\sigma(\theta) = \frac{\sqrt{2\pi}}{nI_n} \int_{\mathbb{R}^n} f(x)\, d\gamma_n(x)\,,$$

where σ is the Haar probability measure on S^{n-1} and γ_n is the standard gaussian probability measure on $\mathbb{R}^n$.

Proof Recall that

$$\mathrm{vol}(B_2^n) = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}\,.$$

(See, for example, [1, Lec. 1].) Thus we compute that

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\, d\gamma_n(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-|x|^2/2}\, dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{S^{n-1}} \int_0^\infty f(r\theta) e^{-r^2/2} r^{n-1}\, dr\, d\theta \\
&= \frac{1}{(2\pi)^{n/2}} \int_{S^{n-1}} \int_0^\infty f(\theta) e^{-r^2/2} r^{n}\, dr\, d\theta \\
&= \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-r^2/2} r^{n}\, dr \int_{S^{n-1}} f(\theta)\, d\theta \\
&= \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-t} (2t)^{(n-1)/2}\, dt \int_{S^{n-1}} f(\theta)\, d\theta \\
&= \frac{\Gamma(1+\frac{n-1}{2})}{\sqrt{2}\pi^{n/2}} \int_{S^{n-1}} f(\theta)\, d\theta \\
&= \frac{1}{\sqrt{2\pi}\,\mathrm{vol}(B_2^{n-1})} \int_{S^{n-1}} f(\theta)\, d\theta \\
&= \frac{n\,\mathrm{vol}(B_2^n)}{\sqrt{2\pi}\,\mathrm{vol}(B_2^{n-1})} \int_{S^{n-1}} f(\theta)\, d\sigma(\theta) \\
&= \frac{nI_n}{\sqrt{2\pi}} \int_{S^{n-1}} f(\theta)\, d\sigma(\theta)
\end{aligned}$$

□

Proposition 4 If X is a standard gaussian random vector in $\mathbb{R}^n$, then

$$\mathbb{E}|X| = \frac{nI_n}{\sqrt{2\pi}}\,,$$

where $|\cdot|$ denotes the Euclidean norm and $\mathbb{E}$ denotes expectation.

Proof By the previous proposition,

$$\int_{\mathbb{R}^n} |x|\, d\gamma_n(x) = \frac{nI_n}{\sqrt{2\pi}} \int_{S^{n-1}} |\theta|\, d\sigma(\theta) = \frac{nI_n}{\sqrt{2\pi}}\,.$$

□

Proposition 5 If P is an orthogonal projection of rank k, then

$$\int_{S^{n-1}} |Px| \, d\sigma(x) = \frac{kI_k}{nI_n} \, .$$

Proof We compute that

$$\begin{aligned}
\int_{S^{n-1}} |P\theta| \, d\sigma(\theta) &= \frac{\sqrt{2\pi}}{nI_n} \int_{\mathbb{R}^n} |Px| \, d\gamma_n(x) \\
&= \frac{\sqrt{2\pi}}{nI_n} \int_{\text{im } P} \int_{\text{ker } P} |P(v + w)| \, d\gamma_{n-k}(w) \, d\gamma_k(v) \\
&= \frac{\sqrt{2\pi}}{nI_n} \int_{\text{im } P} \int_{\text{ker } P} |v| \, d\gamma_{n-k}(w) \, d\gamma_k(v) \\
&= \frac{\sqrt{2\pi}}{nI_n} \int_{\text{im } P} |v| \, d\gamma_k(v) \int_{\text{ker } P} 1 \, d\gamma_{n-k}(w) \\
&= \frac{\sqrt{2\pi}}{nI_n} \cdot \frac{kI_k}{\sqrt{2\pi}} \cdot 1
\end{aligned}$$

□

Proposition 6 For any integer $n \geq 1$,

$$\frac{4^n}{\sqrt{\pi(n + \frac{1}{2})}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}} \, .$$

Proof By induction,

$$I_{2n} = \pi \frac{\binom{2n}{n}}{4^n} \qquad \text{and} \qquad I_{2n+1} = \frac{1}{n + \frac{1}{2}} \cdot \frac{4^n}{\binom{2n}{n}} \, .$$

Since $(I_n)_{n=0}^{\infty}$ is a decreasing sequence, $I_{2n+1} \leq I_{2n} \leq I_{2n-1} = \frac{2n+1}{2n} I_{2n+1}$, that is,

$$\frac{1}{n + \frac{1}{2}} \cdot \frac{4^n}{\binom{2n}{n}} \leq \pi \frac{\binom{2n}{n}}{4^n} \leq \frac{2n+1}{2n} \cdot \frac{1}{n + \frac{1}{2}} \cdot \frac{4^n}{\binom{2n}{n}} \, .$$

Rearranging yields the desired inequalities. □

Proposition 7 (Wallis's product)

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Proof $\frac{\pi}{2} = \frac{I_0}{I_1} = \frac{2}{1} \cdot \frac{I_2}{I_1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{I_2}{I_3} = \dots$ □

References

[1] Keith Ball. An elementary introduction to modern convex geometry. In Silvio Levy, editor, *Flavors of Geometry*, volume 31 of *Mathematical Sciences Research Institute Publications*, pages 1–58. Cambridge UP, Cambridge, 1997. http://www.msri.org/publications/books/Book31/files/ball.pdf. (Cited on page 2.)