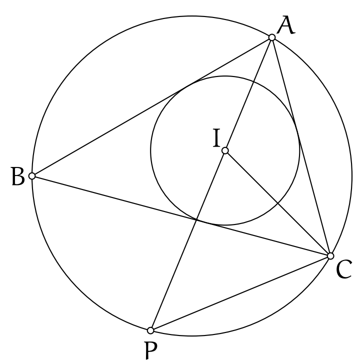
An observation on incentres

Given a triangle $\triangle ABC$, bisect $\angle A$ and extend that bisector to intersect the circumcircle of $\triangle ABC$ at P. Also bisect $\angle C$, and extend that bisector to intersect AP at I. Since I is the intersection of the (internal) angle bisectors, it is the incentre of $\triangle ABC$ (Euclid IV:4).



Join PC. I claim that $PC = PI$. Indeed,

$$\begin{aligned}
\angle PIC &= \angle IAC + \angle ACI && (\angle PIC \text{ an exterior angle of } \triangle IAC\text{; Euclid I:32}) \\
&= \angle PAC + \angle ACI \\
&= \angle PAB + \angle ACI && (AP \text{ bisects } \angle BAC) \\
&= \angle PCB + \angle ACI && (\angle PAB, \angle PCB \text{ stand on same arc; Euclid III:21}) \\
&= \angle PCB + \angle BCI && (CI \text{ bisects } \angle ACB) \\
&= \angle PCI
\end{aligned}$$

and so $PC = PI$ as sides opposite equal angles in $\triangle PCI$ (Euclid I:6).

Exercise: Show that the segment joining the incentre to one of the excentres is bisected by the circumcircle.

Exercise: Construct a triangle, given its inradius, its circumradius, and one of its angles.[1]

Exercise: Given a triangle $\triangle ABC$, bisect each angle and extend those bisectors to intersect the circumcircle of $\triangle ABC$ at points A', B', and C'. Prove that the orthocentre of $\triangle A'B'C'$ is the incentre of $\triangle ABC$.

[1] Problem 1.19.1, George Pólya, *Mathematical Discovery: on understanding, learning, and teaching problem solving* (New York: Wiley, 1981), 185. (Among other places.)